MATRICES AND TRANSFORMATIONS

A 2x2 matrix defines a plane transformation under which the origin is invariant.

A transformation which leaves the origin invariant can be represented by a 2x2 matrix. This means matrices of transformation for reflections in the lines x=0, y=0, y=x and x=-y can be found.

The same applies to the matrices of transformation for rotations about the origin through certain angles.

TRANSFORMING OBJECTS BY MATRICES OF TRANSFORMATION.

-The coordinates of a point (a,b) can be represented by a column matrix as $\binom{a}{b}$.

So if given coordinates of vertices of the object write each in column form in order to form an object coordinates' matrix.

-If the transformation matrix is given, always pre-multiply it by the object coordinates' matrix. The product matrix of the two matrices gives the image coordinates' matrix.

-The coordinates of the image can be got by writing the image coordinates' matrix in coordinate form.

i.e. $\begin{pmatrix} Transformation \\ matrix \end{pmatrix} \begin{pmatrix} coordinates' \ matric \\ for \ object \end{pmatrix} =$

$\begin{pmatrix} coordinates' \ matrix \\ for \ image \end{pmatrix}$

Example 1:

Rectangle ABCD has vertices at A(4,1), B(4,3),C(1,3) and D(1,1).

Find its image under the transformationmatrix$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

Describe the transformation.

Solution:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4 & 4 & 1 & 1\\1 & 3 & 3 & 1\end{pmatrix}=\begin{pmatrix}4 & 4 & 1 & 1\\-1 & -3 & -3 & -1\end{pmatrix}$$

The image rectangle A’B’C’D’ has vertices at A’(4,-1), B’(4,-3), C’(1,-3) and D’(1,-1).

When the image and the object are shown on a Cartesian plane, it indicates that the transformation matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ is a reflection in the x – axis.

Example 2:

A triangle PQR with vertices P(-1,2), Q(2,2) and R(2,4) undergoes a transformationwith matrix $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$to give triangle P’Q’R’.Draw it and its image and describe the transformation.

Solution:

$$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}-1 & 2 & 2\\2 & 2 & 4\end{pmatrix}=\begin{pmatrix}-2 & -2 & -4\\1 & -2 & -2\end{pmatrix}$$

P’(-2,1), Q’(-2,-2), R’(-4,-2).

When the triangles PQR and P’Q’R’ are drawn, it shows that the transformation is a reflection in y=-x

Example 3:

Find the image of the unit square OIKJ under the transformation$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Describe the transformation.

Solution:

Using

$$\begin{pmatrix} Transformation \\ matrix \end{pmatrix}\begin{pmatrix} coordinates' \ matric \\ for \ object \end{pmatrix} = \begin{pmatrix} coordinates' \ matrix \\ for \ image \end{pmatrix}$$

Determine the image square O'I'K'J'. Draw both OIKJ and O'I'K'J' on the same pair of axes, then describe the transformation. It is an enlargement, scale factor 2 and center O(0,0)

Exercise:

Draw the given objects and the corresponding images under the given transformation matrix and hence describe the transformation.

1). Make a capital letter P by joining the points (2,1), (2,2), (2,3), and (3,2). Find the image of the points that make the given letter under the transformation $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

2). Triangle PQR with vertices P(1,1) Q(4,1) and R(4,3). Matrix of transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

3). Rectangle ABCD with vertices A(1,1), B(3,1), C(3,2) and D(1,3); transformation matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

4). Square OIJK with vertices O(0,0), I(1,0), K(1,1) and J(0,1) under matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The unit square

Example:

Examine the effect of the transformation matrix $\begin{pmatrix}2 & 5\\ 1 & 1\end{pmatrix}$ upon the unit square.

Solution:

$$\begin{pmatrix}\mathbf{2} & \mathbf{5}\\ \mathbf{3} & \mathbf{1}\end{pmatrix}\begin{pmatrix}\overset{O}{\mathbf{0}} & \overset{I}{\mathbf{1}} & \overset{K}{\mathbf{1}} & \overset{J}{\mathbf{0}}\\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1}\end{pmatrix} = \begin{pmatrix}\overset{O'}{\mathbf{0}} & \overset{I'}{\mathbf{2}} & \overset{K'}{\mathbf{7}} & \overset{J'}{\mathbf{5}}\\ \mathbf{0} & \mathbf{3} & \mathbf{4} & \mathbf{1}\end{pmatrix};$$

Note: O is invariant, I(1,0) is mapped onto I'(2,3) and that $\begin{pmatrix}2\\ 3\end{pmatrix}$ is the first column in the matrix for the transformation.

What can you notice for J and J'?J(0,1) is mapped onto J'(5,1) and that $\begin{pmatrix}5\\ 1\end{pmatrix}$ is the second column in the matrix of transformation. Therefore, if the matrix of transformation is

A=$\begin{pmatrix}a & c\\ b & d\end{pmatrix}$ then,

$$\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\overset{I}{1} & \overset{J}{0}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\overset{I'}{a} & \overset{J'}{c}\\ b & d\end{pmatrix}$$

$$AI = A$$

Note that, the image coordinates matrix for the unit square gives the matrix of transformation, where the columns for I and J form the **identity matrix** and the columns for I' and J' for the **matrix of transformation.**

What kind of quadrilateral is O'I'K'J'?

A unit square is used to find the matrix of formation because the images of I and J i.e. I' and J' when written in column form, gives the columns of matrix of transformation.

Exercise:

Sketch the image of the unit square under the transformation whose matrix is;

a). $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ b). $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ c). $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ d). $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Describe the transformation in b). and c).

Does a 2x2 always map the unit square on a parallelogram?

Finding the matrix of transformation.

The objects I and J are always used since their coordinates matrix form an identity matrix.

i.e. IA = AI = A.

Example:

If $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$is mapped onto $\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$, find the matrix of transformation.

Solution: The images of I and J are (1,5) and (-1,2) respectively. This means that the columns of the transformation matrix are $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ *and* $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then the matrix $\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$

Unit matrix Image matrix

2). $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 3 \end{pmatrix}$

O' I' K' J'

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$

3). $\begin{pmatrix} 0 & 3 & 7 & 4 \\ 0 & 2 & -1 & -3 \end{pmatrix}$Image under the transformation of the unit square.

Exercise: ref, SMEA BK4 pg84.

1). A triangle ABC with vertices A(1,2), B(-3,5) and C(4,0) is mapped onto a triangle A'B'C' with vertices A'(1,8), B'(-3,9) and C'(4,8). Find the matrix of transformation.

2). Find the matrix of transformation which transforms K(2,1) onto K'(4,5) and L(-3,5) onto L'(-6,-1).

ROTATION

Matrices for rotation

Note: The matrix of transformation for rotation about the origin through the angle ϴ° is given by

$$R=\begin{pmatrix}\cos\theta^\circ & -\sin\theta^\circ \\ \sin\theta^\circ & \cos\theta^\circ\end{pmatrix}$$

i.e. A rotation about the origin through an angle of (i) 90°. (ii)180°, (iii) 270°, (iv) -90. Is

i) $\begin{pmatrix}\cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ\end{pmatrix} = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$

ii) $\begin{pmatrix}\cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ\end{pmatrix} = \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}$

iii) $\begin{pmatrix}\cos 270^\circ & -\sin 270^\circ \\ \sin 270^\circ & \cos 270^\circ\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$

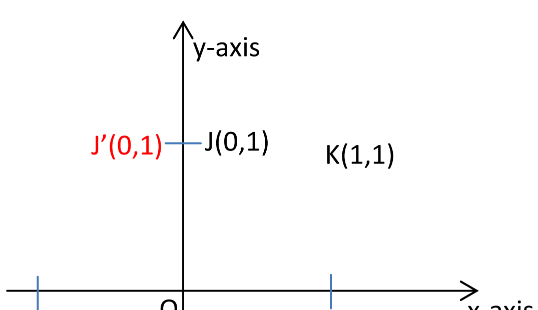
iv) $\begin{pmatrix}\cos -90^\circ & -\sin -90^\circ \\ \sin -90^\circ & \cos -90^\circ\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$

These matrices can also be obtained by rotating the unit square through these angles about the origin. The images of I and J when written in column form give the matrix of transformation.

REFLECTION

Matrices of reflection

if (x, y) is reflected in the y-axis then, (x, y) is mapped onto (-x,y)
In particular,



I(1,0) mapped onto I'(-1,0)

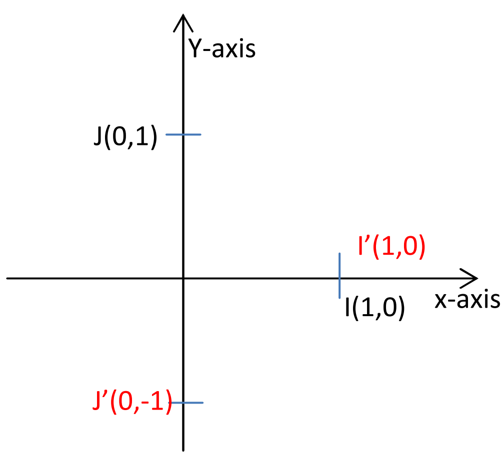
J(0,1) mapped onto J'(0,1)

I J I' J'

A$\begin{pmatrix}1 & 0\end{pmatrix}$ mapped onto $\begin{pmatrix}-1 & 0\end{pmatrix}$

The matrix of transformation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

If (x, y) is reflected on the x –axis, then (x, y) is mapped onto (x, -y)



Similarly, I(1,0) is mapped onto I,(1,0)

J(0,1) is mapped onto J'(0,-1)

The matrix of transformation is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Similarly,

The matrix of transformation for the reflection in y= -x, is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Fory=x, is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

<u>MATRICES OF TRANSFORMATION FOR ENLARGEMENT.</u>

In general the matrix of enlargement scale factor K and centreO(0,0) is

$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$= K$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The shape of the object and its image are similar

Linear ratio is 1:K

Area ratio is $1:K^2$

Therefore the area of the image = K^2 x the area of the object.

Example:

A'B'C' is the image of a triangle ABC with coordinates A(-2,3), B(2,6) and C(5,3) respectively under an enlargement scale factor 5. Show that the ratio of the area of triangle A'B'C' to the area of triangle ABC is 25:1.

Solution

Draw the object and its image on a Cartesian plan and find their areas.

AREA AND DETERMINANTS

When the object and matrix of transformation are known, we can find the area of the image using the area of the object and the determinant.

Example:

Rectangle O(0,0), A(2,0), B(2,1), C(0,1) has been transformed through the following matrices of transformation.

i) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$ iii) $\begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}$ iv) $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

Use, Area of image = determinant x area of object.

Therefore, Determinant = Area scale factor.

Note: The determinant can be negative but its absolute value is used.

Example:

A rectangle with vertices (0,0), (2,0), (2,3), (0,3) is transformed using matrix $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

i).Find the coordinates of the vertices of the image.

ii). Sketch the object and the image, Find the area of the image.

Solution:

Object matrix = $\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 9 & 3 \\ 0 & 2 & 5 & 3 \end{pmatrix}$$

The vertices of the image (0,0), (6,2), (9,5) and (3,3).

ii) Area of the object = 2x3=6 square units.

Determinant of the matrix = 3x1 – 1x1 =2.

Therefore the area scale factor is 2.

Therefore area of the image = 2x6=12 square units.

Example:

Find the image of a unit a square under the transformation given by$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$. Draw the diagram for the object and image and use it to find the area scale factor.

Solution:

From the diagram, image area = 2square units.

The area scale factor = 2 but determinant = -2.

N.B

The area scale factor is equal to the magnitude of the determinant.

Exercise:

The area of an object is 4cm^2. State the area of the image under the transformations given by the following matrices.

i) $\begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$ ii) $\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$ iii) $\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ iv) $\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$.

SPECIAL CASES

a). When the determinant of a matrix of transformation is 1, the area scale factor is 1. So, the area of an object is the same as the area of its image. Transformations in which the area is invariant include; rotations, reflections, shears and translations. However, translations cannot be represented by a 2x2 matrix.

Example:

1). A transformation is given$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

What is the area scale factor of a unit square under this transformation.

What are the coordinates of the image?

2). Find the area of the image of a unit square under a transformation given by $\begin{pmatrix} 1 & 3 \\ \frac{1}{3} & 1 \end{pmatrix}$..

Solution:

Determinant = 1x1 - 3x$\frac{1}{3}$ = 0

Therefore, Area scale factor = 0

In fact the image is a line segment, so the area is 0.

Shearing:

The determinant of the matrix for any shear is 1, since the area property of shearing is that it is invariant.

The matrix of a shear with the x-axis invariant is, $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

With the y-axis, invariant is $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

N.B Sketch the image of the unit square when the matrix of transformation is $\begin{pmatrix} 1 & 3 \\ 0 & \frac{1}{3} \end{pmatrix}$.

Explain from the figure why this transformation is not a shear.
What is the area of the image?
What is the value of the determinant?

Solution:

The x-axis is an invariant line, but the points J and K have not moved parallel to this invariant line.

The area of the image = $\frac{1}{3}$ sq. units

The determinant is $\frac{1}{3}$ and so the area scale factor is $\frac{1}{3}$thus the area is not invariant as it would be under a shear.

Note:

If the x-axis is invariant, for any shear, then the points J and K have to move parallel to this invariant line. Similarly, if the y-axis is invariant, the points J and K have to move parallel to this line.

Transformation	**matrix**	**symbol**
Positive quarter turn about O	R=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	R
Half turn about O	H=$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	H
Negative quarter turn about O	Q=$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Q
Reflection in x-axis	X=$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	X
Reflection in y-axis	Y=$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Y

Reflection in line y=x	M=$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	M
Reflection in line y=-x	N=$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	N
Identity	I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	I

IMAGES OF LINES UNDER A MATRIX OF TRANSFORMATION.

A line is a set of points so, to find its image under a given matrix of transformation, get at least two points on the line. Use the points to form a coordinates matrix.

Examples:

Find the images of the lines;

i) 3x + 2y = -6

ii)Y = 4x + 12

Under the following transformations.

a). $^1/_4$ turn, center (0,0).

b). a reflection in line y=x.

sketch the objects and their images on the same diagram.

Exercise:

1). Find the image of the line y=-2x + 4 when reflected in the x-axis then in the line y=x. Sketch the two lines.

2). Sketch the image of the line 2y + 5x = 10, when it is reflected in the line y + x = 0; then rotated through a positive turn about the origin.

END OF TRANSFORMATION